

DEGENERATION AND CURVES ON K3 SURFACES

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ABSTRACT. This paper studies curves on quartic K3 surfaces, or more generally K3 surfaces which are complete intersection in weighted projective spaces. A folklore conjecture concerning rational curves on K3 surfaces states that all K3 surfaces contain infinite number of irreducible rational curves. It is known that all K3 surfaces, except those contained in the countable union of hypersurfaces in the moduli space of K3 surfaces satisfy this property. In this paper we present a new approach for constructing curves on varieties which admit nice degenerations. We apply this technique to the above problem and prove that there is a Zariski open dense subset in the moduli space of quartic K3 surfaces whose members satisfy the conjecture. Various other curves of positive genus can be also constructed.

1. INTRODUCTION

This paper studies curves on quartic K3 surfaces, or more generally K3 surfaces which are complete intersection in weighted projective spaces. Studying curves on K3 surfaces is a subject of constant research, both from the points of view of classical algebraic geometry and more recent study motivated by physics.

A folklore conjecture concerning rational curves on K3 surfaces is the following (see [9, Section 13] for more details about the historical development related to this conjecture).

Conjecture 1. Every polarized K3 surface (X, H) over an algebraically closed field contains infinitely many integral rational curves linearly equivalent to some multiple of H .

Mori and Mukai [15] showed (attributed to Bogomolov and Mumford) that every complex polarized K3 surface (X, H) contains at least 1 rational curve which belongs to the linear system $|H|$. One subtle point about this problem is the difference between *generic* and *general*. Here generic means a property held by members in a non-empty Zariski open subset of the moduli space of K3 surfaces. While general means a property held by members in the complement of the countable union of proper Zariski closed subsets (typically hypersurfaces). Using this terminology, one can deduce from Mori and Mukai's argument that a general polarized complex K3 surface (X, H) contains infinitely many irreducible rational curves linearly equivalent to some nH (see [9, Corollary 1.2, Section 13]).

In the late 90's, Chen [3] proved the existence of infinitely many irreducible nodal rational curves on general K3 surfaces. More recently, Bogomolov-Hassett-Tschinkel [2] and Li-Liedtke [14] proved stronger result for general K3 surfaces based on sophisticated arithmetic geometric argument as well as deep geometry of K3 surfaces.

Our method in this paper is essentially elementary, constructive, and does not much depend on the special properties of K3 surfaces. In fact, it can also be used to produce many kinds of holomorphic curves in variety of situations, see Subsection 7.1.

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There are some unconventional points in our arguments, which might confuse the readers at the first glance. Among those points are:

- We make use of log smooth deformation theory ([10, 12]), but the log structures we put on K3 surfaces will not be defined on the whole surface. Since we are interested in the deformation of curves on K3 surfaces, it suffices to put log structures only on a suitable neighborhood of the curve, see Remark 7.
- To see the existence of a smoothing of a degenerate rational curve, we must prove the vanishing of the obstruction. To prove this vanishing of the obstruction for rational curves, we make use of the vanishing of them for higher genus curves, which is almost immediate. See Section 3 and the beginning of Section 4.

The basic idea in this paper to construct rational curves is the same as in [3, 2, 14], although the original motivation came from a different perspective (tropical geometry). Namely, we consider a degenerating family of K3 surfaces and construct degenerate rational curves on the central fiber of it. Then prove that the degenerate curves can be deformed into a generic fiber of the family.

The difference in the construction here compared to the previous approach is that the degeneration we consider has the fixed central fiber (the union of 4 projective planes). This is contrary to the situation of [3] where the central fiber is the union of two rational surfaces intersecting along a general elliptic curve, and one may have to change the central fiber when he wants to obtain curves of different degrees (however, it is quite plausible that the argument in this paper can be also applied to this kind of degenerations).

In our situation, varying the defining polynomials of the quartic K3 surfaces we start from is reflected to the distribution of the singular locus of the total space of the degeneration on the central fiber. In other words, fixing the defining equation determines the direction along which the degenerating family approaches to the fixed degenerate fiber. Then the main problem is to prove that for Zariski open subset in the space of quartic homogeneous polynomials, infinitely many degenerate rational curves can be deformed into a generic fiber. This will be done by proving the vanishing of the obstruction in Section 6. As noted above, we prove the vanishing by comparing obstructions of curves with different genus. This point is discussed in Sections 3 and 5. In Section 4, we construct infinitely many degenerate rational curves suitable to our argument.

As we noted above, our argument does not much depend on the geometry of K3 surfaces. For example, the same ideas in this paper can be applied to construct rational curves on Calabi-Yau quintic 3-folds and holomorphic disks on hypersurfaces on \mathbb{P}^n ([19]), and holomorphic curves on Abelian surfaces ([22]). Also, in a subsequent paper, we will use the idea of this paper to prove denseness result of rational curves on generic (complete intersection) K3 surfaces, which will lead to the solution of the full Conjecture 1 (see Section 7).

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2. DEGENERATION OF K3 SURFACES AND OBSTRUCTION

In this section, we recall some calculations in [19]. There we considered a degeneration of a quartic K3 surface

$$xyzw + tf(x, y, z, w) = 0,$$

where x, y, z, w are homogeneous coordinates of \mathbb{P}^3 , $t \in \mathbb{C}$ is the parameter of the degeneration and f is a generic homogeneous quartic polynomial of x, y, z, w . Let

$$\mathfrak{X} \subset \mathbb{P}^3 \times \mathbb{C}$$

be the variety defined by the above equation (the total space of the degeneration), and X_0 be the central fiber. Let

$$i_0: X_0 \rightarrow \mathfrak{X}$$

be the inclusion. The space X_0 is the union of 4 projective planes glued along projective lines:

$$X_0 = \cup_{i=1}^4 \mathbb{P}_i^2$$

Each \mathbb{P}_i^2 has a natural structure of a toric variety, in which the lines mentioned above are the toric divisors. Let

$$\ell_1, \dots, \ell_6$$

be these projective lines. Let

$$L = \cup_{i=1}^6 \ell_i$$

be the union of them. Each ℓ_i has 2 distinguished points, which are the triple-intersections of the projective planes. We write by ℓ_i° the complement of these 2 points.

Since f is generic, the total space \mathfrak{X} has 24 singular points, and for each i , 4 of them lie on ℓ_i° . Let

$$\mathcal{S} \subset X_0$$

be the set of these singular points.

We consider the following problem. Let C_0 be a pre-stable curve and

$$\varphi_0: C_0 \rightarrow X_0$$

be a stable map. Then the problem is, whether the map φ_0 has a smoothing or not. More precisely, we ask whether there is a family of pre-stable curves

$$\mathcal{C} \rightarrow \mathbb{C}$$

such that the fiber over 0 is C_0 , and a family of stable maps

$$\Phi: \mathcal{C} \rightarrow \mathfrak{X}$$

over \mathbb{C} which coincides with φ_0 on C_0 .

2.1. Pre-log curves and pre-smoothable curves. As is pointed out in [13] (see also [21] in the context of tropical curves), for an immersed stable map $\varphi_0: C_0 \rightarrow X_0$, if the image is away from the singularities of \mathfrak{X} , it is necessary for φ_0 to satisfy the *pre-log* condition (Definition 4.3, [21]) to solve the above problem. We now recall the pre-log condition.

Definition 2 ([21, Definition 4.1]). Let Y be a toric variety. A holomorphic curve $C \subset Y$ is *torically transverse* if it is disjoint from all toric strata of codimension greater than 1. A stable map $\phi: C \rightarrow Y$ is torically transverse if $\phi^{-1}(\text{int}Y) \subset C$ is dense and $\phi(C) \subset Y$ is a torically transverse curve. Here $\text{int}Y$ is the complement of the union of toric divisors.

Let $\pi: \mathfrak{Y} \rightarrow \mathbb{C}$ be a flat family of varieties such that

- The fibers $Y_t = \pi^{-1}(t)$, $t \neq 0$ are nonsingular irreducible varieties.

- The special fiber Y_0 is a union of toric varieties $Y_0 = \cup_{i=1}^k Y_{0,i}$ such that for any different $i, j \in \{1, \dots, k\}$, the intersection $Y_{0,i} \cap Y_{0,j}$ is a toric stratum of both $Y_{0,i}$ and $Y_{0,j}$.
- For each point $p \in Y_0$, there is an analytic neighborhood $U \subset \mathfrak{Y}$ with the property that U is analytically isomorphic to an open subset of a toric variety and the restriction $U \rightarrow \mathbb{C}$ of π to U has a natural (up to isomorphisms) structure of an open subset of a degeneration of toric varieties whose central fiber is $U \cap Y_0$.

In other words, any point on Y_0 has a neighborhood modeled on a toric degeneration of toric varieties.

Definition 3. Let C_0 be a prestable curve. A *pre-log curve* on Y_0 is a stable map $\varphi_0: C_0 \rightarrow Y_0$ with the following properties.

- For any component $Y_{0,i}$ of Y_0 , the restriction $C \times_{Y_0} Y_{0,i} \rightarrow Y_{0,i}$ is a torically transverse stable map.
- Let $p \in C_0$ be a point which is mapped to the singular locus of Y_0 . Then C_0 has a node at p , and φ_0 maps the two branches $(C'_0, p), (C''_0, p)$ of C_0 at P to different irreducible components $Y_{0,i'}, Y_{0,i''} \subset Y_0$. Moreover, if w' is the intersection index of the restriction $(C'_0, p) \rightarrow (Y_{0,i'}, D')$ with the toric divisor $D' \subset Y_{0,i'}$, and w'' accordingly for $(C''_0, p) \rightarrow (Y_{0,i''}, D'')$, then $w' = w''$.

Let \mathcal{T} be the singular locus of the total space \mathfrak{Y} . Note that \mathcal{T} is contained in the singular locus (in particular, in the union of toric strata of positive codimension) of Y_0 . Then we generalize the notion of pre-log curves to this case (in [21], the total space \mathfrak{Y} itself is toric and the singular locus \mathcal{T} can be assumed to be empty).

As we noted above, if the image of a map $\varphi_0: C_0 \rightarrow Y_0$ intersects a singular point of Y_0 not contained in \mathcal{T} , the map φ_0 must satisfy the pre-log condition above for the existence of smoothings. In particular, if a smooth point of C_0 is mapped to a toric boundary of a component of Y_0 , then it must be mapped into the singular subset \mathcal{T} . We record this point as a lemma.

Lemma 4. Let φ_0 be as above. Also assume the map φ_0 admits a smoothing Φ in the above sense. Let $x \in C_0$ be a smooth point which is mapped to a toric boundary of a component of Y_0 by φ_0 . Then, the image $\varphi_0(x)$ must be contained in the set \mathcal{T} . \square

We give a slight generalization of the notion of pre-log curves, to account for the singularity of the total space.

Definition 5. We call a map $\varphi_0: C_0 \rightarrow Y_0$ *pre-smoothable* if it satisfies the following conditions.

- The set $\varphi_0^{-1}(\mathcal{T})$ is a finite set consisting of regular points of C_0 .
- The restriction of the map φ_0 to $C_0 \setminus \varphi_0^{-1}(\mathcal{T})$ is a pre-log curve.

In our situation, the degeneration $\mathfrak{X} \rightarrow \mathbb{C}$ satisfies the above condition for $\pi: \mathfrak{Y} \rightarrow \mathbb{C}$. By Lemma 4, if the map φ_0 admits a smoothing, then it must be pre-smoothable. The curves we study are rather simple ones among pre-smoothable curves.

Definition 6. Let $\varphi_0: C_0 \rightarrow X_0$ be a pre-smoothable curve. If the map φ_0 satisfies the condition

- the image of any irreducible component of C_0 by φ_0 is a curve of degree 1 in some component of $X_0 = \cup_{i=1}^4 \mathbb{P}_i^2$,

then we call φ_0 *simply pre-smoothable*.

2.2. Log structures on a neighborhood of a curve and log normal sheaves of simply pre-smoothable curves. Now we recall the calculation in [19] of the sheaves which control the deformation of φ_0 . This uses the notion of log structures. Let \mathbb{P}_i^2 be a component of X_0 . Let ℓ be one of the toric divisors of \mathbb{P}_i^2 . Let $x \in \ell$ be a point on ℓ° which is not contained in the set \mathcal{S} . Then there is a neighbourhood of the point x in \mathfrak{X} which is isomorphic to a neighbourhood of the origin of the variety defined by the equation

$$z_1 z_2 + t = 0$$

in $\mathbb{C}^3 \times \mathbb{C}$ with coordinates (z_1, z_2, z_3, t) . This variety has a natural structure of a toric variety over $\mathbb{C} = \text{Spec } \mathbb{C}[t]$ (which is also seen as a toric variety) and we put a natural log structure coming from the toric structure. A chart for this log structure is given by

$$\begin{aligned} \mathbb{N}\langle \tau_1, \tau_2, \tau_t \rangle / (\tau_1 + \tau_2 - \tau_t) &\rightarrow \mathbb{C}[z_1, z_2, t] / (z_1 z_2 + t), \\ \tau_1 &\mapsto z_1, \quad \tau_2 \mapsto z_2, \quad \tau_t \mapsto -t. \end{aligned}$$

Here $\mathbb{N}\langle \tau_1, \tau_2, \tau_t \rangle$ is the monoid generated by τ_1, τ_2 and τ_t . Note that the ghost sheaf for this log structure is isomorphic to $\mathbb{N}\langle \bar{\tau}_1, \bar{\tau}_2, \bar{\tau}_t \rangle / (\bar{\tau}_1 + \bar{\tau}_2 - \bar{\tau}_t)$, here $\bar{\tau}_i$ is the image of τ_i .

On the other hand, if $x \in \ell$ lies in the set \mathcal{S} , then \mathfrak{X} is locally isomorphic to a neighborhood of the origin of the variety defined by the equation

$$z_1 z_2 + t z_3 = 0.$$

This also has a natural toric structure, and we put a log structure coming from it. A chart for this log structure is given by

$$\begin{aligned} \mathbb{N}\langle \tau_1, \tau_2, \tau_3, \tau_t \rangle / (\tau_1 + \tau_2 - \tau_3 - \tau_t) &\rightarrow \mathbb{C}[z_1, z_2, z_3, t] / (z_1 z_2 + t z_3), \\ \tau_1 &\mapsto z_1, \quad \tau_2 \mapsto z_2, \quad \tau_3 \mapsto z_3, \quad \tau_t \mapsto -t. \end{aligned}$$

Note that the ghost sheaf for this log structure is isomorphic to $\mathbb{N}\langle \bar{\tau}_1, \bar{\tau}_2, \bar{\tau}_3, \bar{\tau}_t \rangle / (\bar{\tau}_1 + \bar{\tau}_2 - \bar{\tau}_3 - \bar{\tau}_t)$.

Remark 7. As remarked in [19, Remark 5], it is important to notice that we do not need to put a log structure on whole \mathfrak{X} , but only around the curve $\varphi_0(C_0)$. The local log structures above may not extend globally to \mathfrak{X} , but it does not matter in our argument. We put log structures around the intersection of $\varphi_0(C_0)$ and the toric divisors of the components of X_0 in the way described above, and around other points of $\varphi_0(C_0)$, we put a strict log structure pulled back from the standard log structure on \mathbb{C} as a toric variety. This is possible when the subvariety of \mathfrak{X} locally defined by the equation $z_3 = 0$ in the above notation around the singular locus of \mathfrak{X} intersects the image $\varphi_0(C_0)$ transversally. For general f (the defining polynomial of the degenerating K3 surface), we can always assume this condition in the argument below.

We put a log structure on C_0 over $\text{Spec } \mathbb{C}[t]$ which is described in [21, Proposition 7.1] on neighborhoods of the nodes, and in [19, Section 2.1] on neighborhoods of points in the inverse image of the set \mathcal{S} , and put a strict log structure over $\text{Spec } \mathbb{C}[t]$ otherwise. Precisely, a chart around a node x_0 is given by

$$\begin{aligned} \mathbb{Z}\langle \sigma_S, \sigma_T, \sigma_t \rangle / (\sigma_S + \sigma_T - \sigma_t) &\rightarrow \mathcal{O}_{C_0}|_{U_{x_0}}, \\ \sigma_S &\mapsto S, \quad \sigma_T \mapsto T, \quad \sigma_t \mapsto 0. \end{aligned}$$

Here U_{x_0} is a neighborhood of the node x_0 and S, T are coordinates on the branches of C_0 at x_0 . A chart around a point x_1 in the inverse image of the set \mathcal{S} is given by

$$\begin{aligned} \mathbb{N}\langle\sigma_S, \sigma_t\rangle &\rightarrow \mathcal{O}_{C_0}|_{U_{x_1}}, \\ \sigma_S &\mapsto S, \quad \sigma_t \mapsto 0. \end{aligned}$$

Here U_{x_1} is a neighborhood of the point x_1 and S is a coordinate on C_0 which is 0 at x_1 .

Then the map φ_0 can be equipped with a structure of a map between log schemes with the following properties. Note that the ghost sheaf on U_{x_0} is isomorphic to a monoid $\mathbb{N}\langle\bar{\sigma}_S, \bar{\sigma}_T, \bar{\sigma}_t\rangle/(\bar{\sigma}_S + \bar{\sigma}_T - \bar{\sigma}_t)$ and the ghost sheaf on U_{x_1} is isomorphic to a monoid $\mathbb{N}\langle\bar{\sigma}_S, \bar{\sigma}_t\rangle$.

- The composition of φ_0 with the projection to $\text{Spec } \mathbb{C}[t]$ is log smooth.
- The map φ_0 is strict away from the nodes and the inverse image of the singular set \mathcal{S} of \mathfrak{X} .
- The map φ_0 induces a map between ghost sheaves around a node by

$$\bar{\tau}_1 \mapsto \bar{\sigma}_S, \quad \bar{\tau}_2 \mapsto \bar{\sigma}_T, \quad \bar{\tau}_t \mapsto \bar{\sigma}_t.$$

- The map φ_0 induces a map between ghost sheaves around the inverse image of \mathcal{S} by

$$\bar{\tau}_1 \mapsto \bar{\sigma}_S, \quad \bar{\tau}_2 \mapsto 0, \quad \bar{\tau}_3 \mapsto \bar{\sigma}_S, \quad \bar{\tau}_t \mapsto \bar{\sigma}_t.$$

The structure as a map between log schemes on φ_0 satisfying these conditions is uniquely determined up to isomorphisms.

Log smooth deformations of φ_0 are controlled by its log normal sheaf $\mathcal{N}_{\varphi_0} = \varphi_0^* \Theta_U / \Theta_{C_0}$. Here U is a neighborhood of $\varphi_0(C_0)$ in \mathfrak{X} (see Remark 7 above), Θ_U and Θ_{C_0} are the log tangent sheaves with respect to the above log structures. In [19, Subsection 2.1.1], we computed this sheaf for the case of a line on X_0 . The computation for general pre-smoothable curves with arbitrary genus is done by gluing the restrictions of \mathcal{N}_{φ_0} to irreducible components of C_0 , which reduces the computation to the case of a line above. So the computation is essentially the same as that in [19, Subsection 2.1.1] and we obtain the following.

Proposition 8. *Let $\varphi_0: C_0 \rightarrow X_0$ be a simply pre-smoothable curve. Then the log normal sheaf \mathcal{N}_{φ_0} is an invertible sheaf, and if $C_{0,i}$ is a component of C_0 , then the restriction of \mathcal{N}_{φ_0} to $C_{0,i}$ is isomorphic to*

$$\mathcal{N}_{C_{0,i}/\mathbb{P}^2}(-\sum_i x_i),$$

here $\mathcal{N}_{C_{0,i}/\mathbb{P}^2}$ is the usual (non log) normal sheaf of $\varphi_0|_{C_{0,i}}$ as a map to a component ($\cong \mathbb{P}^2$) of X_0 , and the set $\{x_i\}$ is inverse image of the intersection $\varphi_0|_{C_{0,i}}(C_{0,i}) \cap \mathcal{S}$. \square

The Zariski tangent space of the space of log-smooth deformations of φ_0 is given by

$$H^0(C_0, \mathcal{N}_{\varphi_0})$$

and the obstruction is given by

$$H^1(C_0, \mathcal{N}_{\varphi_0}).$$

Lemma 9. *Let $\varphi_0: C_0 \rightarrow X_0$ be a simply pre-smoothable curve. Then we have the isomorphism*

$$H^1(C_0, \mathcal{N}_{\varphi_0}) \cong \mathbb{C}.$$

Proof. This follows from the results in [19, Section 2]. We briefly recall the description.

Using Serre duality for nodal curves, we actually calculate the dual group

$$H^1(C_0, \mathcal{N}_{\varphi_0})^\vee \cong H^0(C_0, \mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0}),$$

where ω_{C_0} is the dualizing sheaf on C_0 . A section of the sheaf $\mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0}$ is described as follows. Namely, let

$$\Sigma \subset N_2 \otimes \mathbb{R} \cong \mathbb{R}^2$$

be a fan whose associated toric variety is \mathbb{P}^2 . Here N_2 is the free abelian group of rank 2. Let

$$e_1, e_2, e_3 \in N_2$$

be the primitive integral generators of the rays of Σ . Let

$$f_1, f_2, f_3 \in N_2^\vee$$

be the vectors of the dual lattice of N_2 which annihilate the vectors e_1, e_2, e_3 respectively. These vectors can be taken so that

$$f_1 + f_2 + f_3 = 0.$$

Then a section of $\mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0}$ restricted to a component $C_{0,i}$ is given by an $N_2^\vee \otimes \mathbb{C}$ -valued meromorphic 1-form η with the following properties:

- The 1-form η can have poles only at the points of $C_{0,i}$ which are nodes of C_0 , or included in the inverse image $\varphi_0^{-1}(\mathcal{S})$. In particular, for any i , η can have poles at 3 points, and the orders of them are at most 1.
- Let r_1, r_2, r_3 be the $N_2^\vee \otimes \mathbb{C}$ -valued residues at the poles. One can see that (after renumbering if necessary) these must be of the form

$$r_1 = c_1 f_1, \quad r_2 = c_2 f_2, \quad r_3 = c_3 f_3$$

for some complex numbers c_1, c_2, c_3 . Moreover, these vectors satisfy

$$r_1 + r_2 + r_3 = 0.$$

Note that $r_1 + r_2 + r_3 = 0$ if and only if

$$c_1 = c_2 = c_3.$$

In particular, the residue r_1 at a pole determines the residues at the other 2 poles.

Now let $C_{0,i}$ and $C_{0,j}$ be neighboring components of C_0 and let $x_{i,j} = C_{0,i} \cap C_{0,j}$. Then sections η_i, η_j of $\mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0}$ restricted to these components glue into a section of $\mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0}$ over $C_{0,i} \cup C_{0,j}$ if and only if the residues r_i, r_j of η_i, η_j at $x_{i,j}$ satisfy

$$(*) \quad r_i + r_j = 0.$$

Using these relations, one sees that when one fixes a residue r at a node of C_0 or at a point $x \in \varphi_0^{-1}(\mathcal{S})$, then there is a unique global section of $\mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0}$ whose residue at x is r . Thus, the group $H^0(C_0, \mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0})$ (and so the group $H^1(C_0, \mathcal{N}_{\varphi_0})$) is 1 dimensional. \square

Remark 10. Note that Lemma 9 applies regardless of the genus of C_0 . Also note that each component of C_0 is a rational curve with 3 special points (that is, the points mapped to the toric divisors of the components of X_0), and so belongs to the unique isomorphism class. Given 2 simply pre-smoothable curves $\varphi_0: C_0 \rightarrow X_0$ and $\varphi'_0: C'_0 \rightarrow X_0$, consider components $C_{0,v}$ and $C_{0,v'}$ of C_0 and C'_0 which are mapped to the same component of

X_0 . Then under this isomorphism, a generator of the groups $H^0(C_0, \mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0})$ and $H^0(C'_0, \mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C'_0})$ restricts to the same meromorphic 1-form (up to a constant multiple) on $C_{0,v}$ and $C_{0,v'}$. Here the curves C_0, C'_0 need not be isomorphic (in particular, the genera of them may be different).

We also need to consider the case where the stable map is not pre-smoothable. The main example of such a curve is a limit of degenerate higher genus curves, see Subsection 3.2. An important point is that the image of such a curve is the same as the image of a map from a degenerate rational curve. Another important point is that although such a map may not fit in the formalism of log smooth deformation theory, the generator of $H^0(C_0, \mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0})$ makes sense (see Subsection 3.3.1).

3. CALCULATION OF THE OBSTRUCTION IN THE SIMPLEST CASE AND THE EXISTENCE OF THE SMOOTHING

3.1. Remarks on calculation of the obstruction. Given a simply pre-smoothable curve $\varphi_0 : C_0 \rightarrow X_0$, we calculated the generator η of the dual obstruction class $H^0(C_0, \mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0})$ in Lemma 9. On the other hand, the obstruction class in $H^1(C_0, \mathcal{N}_{\varphi_0})$ is calculated as follows.

Namely, we can take a suitable covering $\{U_i\}_i$ of C_0 so that on U_i there is a local lift of the map φ_0 . Such a covering exists since the projection $\mathfrak{X} \rightarrow \mathbb{C}$ is log smooth. These local lifts are torsor over the group of sections of \mathcal{N}_{φ_0} , and so the difference of the lifts on the intersections $U_i \cap U_j$ defines a \mathcal{N}_{φ_0} -valued Čech 1-cocycle. This is the obstruction class in $H^1(C_0, \mathcal{N}_{\varphi_0})$.

It is clear that the class does not depend on the choices of local lifts, since by construction the cocycles defined by two lifts differ only by coboundary. It is not easy in general to directly identify the obstruction cohomology class (in particular, determine whether it is zero or nonzero) from the presentation as a Čech cohomology class. More efficient way is to calculate the coupling between $H^1(C_0, \mathcal{N}_{\varphi_0})$ and its dual $H^0(C_0, \mathcal{N}_{\varphi_0}^\vee \otimes \omega_{C_0})$, and it can be reduced to the calculation of appropriate residues at the nodes, see [19].

In this paper we do not really need to do the actual calculation of the obstruction class since we can show that it vanishes by another reason. However, since the actual calculation will help understand the later argument, we perform a calculation of the obstruction in a simple case in the next subsection.

3.2. Degenerate curves of degree 4 and their obstructions. Our construction of rational curves on a quartic K3 surface will be done by deforming a rational curve on the degenerate space X_0 to general fibers X_t , $t \neq 0$. So the starting point is the construction of a rational curve on X_0 . This can be done by considering intersections with other particular surfaces.

Let us start from the simpler case of general curves of degree 4. Such a curve on a quartic K3 surface or its degeneration is obtained by taking the intersection with a general hyperplane H . In particular, we assume that the image of the map $\varphi_0 : C_0 \rightarrow X_0$ does not intersect the singular locus \mathcal{S} of \mathfrak{X} . Thus, the map φ_0 is a pre-log curve in the sense of [21]. On the degenerate space X_0 , the intersection will be the union of 4 lines (see Figure 1).

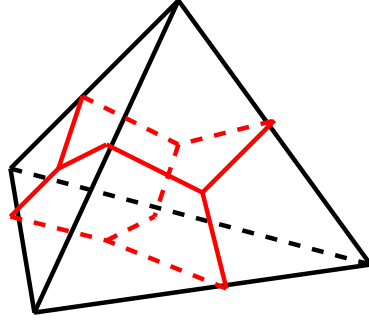


FIGURE 1. Picture of a general curve of degree 4 on X_0 . Here X_0 is the union of 4 \mathbb{P}^2 s. The tetrahedron is the intersection complex of X_0 , so each triangle corresponds to \mathbb{P}^2 , and the triangle can be regarded as the moment polytope of \mathbb{P}^2 . On the other hand, the graph on the tetrahedron is the *dual* intersection graph of a curve on X_0 .

Now we turn to the construction of degenerate rational curves. As we mentioned in Section 2, the total space \mathfrak{X} of the degeneration contains the singular set \mathcal{S} consisting of 24 points. On the tetrahedron of Figure 2, every edge contains 4 of these singular points.

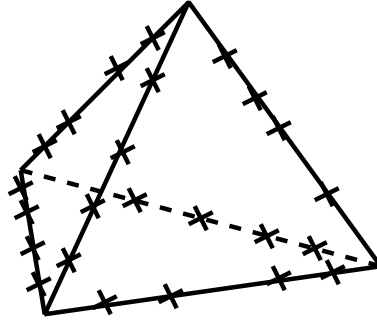


FIGURE 2. The cross marks mean the singular points of the total space \mathfrak{X} of the degeneration.

If we choose a hyperplane H so that H intersects 3 singular points which are not contained in a single component of X_0 , we can regard the intersection of H and X_0 as the image of a map from a nodal rational curve, and by Lemma 4, we can ask whether this map can be lifted to a non-zero fiber X_t , $t \neq 0$, giving a rational curve there, see Figure 3. Let us write by

$$\psi_0: B_0 \rightarrow X_0$$

the stable map from a nodal rational curve to X_0 obtained in this way.

On the other hand, we can also choose a family of hyperplanes H_s , $s \in D$ parametrized by a small disk D around the origin of \mathbb{C} whose central fiber H_0 is the hyperplane H chosen above. The intersection between $H_s \times \mathbb{C} \subset \mathbb{P}^3 \times \mathbb{C}$ and \mathfrak{X} gives a family of degenerating families of curves of degree 4. In particular, the intersection between $H_0 \times \mathbb{C}$ and \mathfrak{X} also gives a degenerating family whose general fiber is a smooth curve of genus 3. Thus, the image $\psi_0(B_0)$ can also be seen as a degenerate genus 3 curve. We write it by

$$\varphi_{0,0}: C_{0,0} \rightarrow X_0,$$

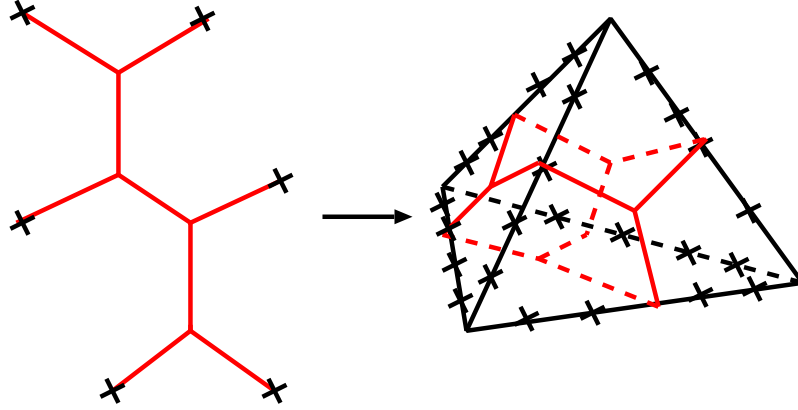


FIGURE 3. Picture of a degenerate rational curve $\psi_0(B_0)$ on X_0 . The picture on the left is the dual intersection graph of the nodal rational curve which is the domain of the map ψ_0 . The cross marks on the graph corresponds to regular points on the rational curve mapped to the singular locus of \mathfrak{X} .

see Figure 4. We also write by

$$\varphi_{0,s} : C_{0,s} \rightarrow X_0$$

the degenerate curve obtained as the intersection between H_s and X_0 . Note that all the domain curves $C_{0,s}$ are in fact isomorphic, including $C_{0,0}$.

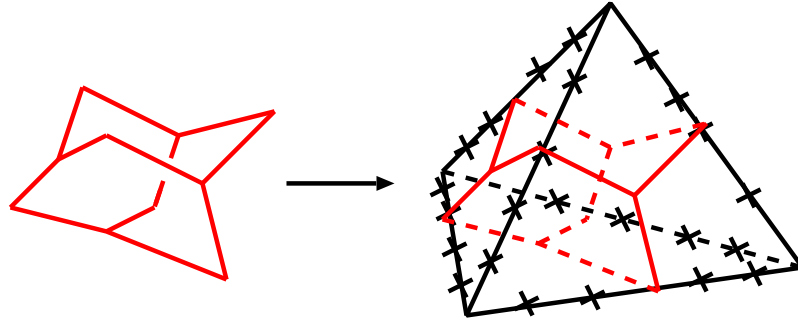


FIGURE 4. The intersection $H \cap X_0$ seen as a degenerate genus 3 curve $\varphi_{0,0}(C_{0,0})$.

Compared with the map $\psi_0 : B_0 \rightarrow X_0$ above, the map $\varphi_{0,0}$ has the different domain curve, but their images are the same. We do not calculate the obstruction cohomology group of the map $\varphi_{0,0}$ (see Remark 18). However, the generator η of the obstruction to lift the maps $\varphi_{0,s}$, $s \neq 0$ calculated in Lemma 9 can be naturally defined on $C_{0,0}$, since all $C_{0,s}$ are isomorphic (see Remark 10).

Also, we can construct a family of obstruction Čech 1-cocycles on $C_{0,s}$ by using a family version of local lifts considered in Subsection 3.1, parametrized by s . These cocycles couple with the above generator η , and the result is an analytic function of s . This analytic function is identically 0.

Remark 11. (1) For $s \neq 0$, the Čech 1-cocycle above takes values in $\varphi_{0,s}^* \Theta_{X_0}$, where the log structure on X_0 is given by the one from the isomorphism with the toric

variety $z_1 z_2 + t = 0$ (see Subsection 2.2). However, for $s = 0$, the Čech 1-cocycle defined in the above way may not necessarily take values in the same sheaf (see Subsection 3.3.1), because due to the singularity of \mathfrak{X} with which $\varphi_{0,0}$ intersects, the log structure of a neighborhood of the image of $\varphi_{0,s}$ changes. Nevertheless, since all the domain curves $C_{0,s}$ can be naturally identified and the family of Čech 1-cocycles is analytic with respect to s , any explicit way of the calculation of the coupling between the Čech 1-cocycle and η for $s \neq 0$ naturally extends to $s = 0$, thus the coupling for $s = 0$ also makes sense.

- (2) It is obvious that the resulting analytic function of s is identically 0, since one of families of lifts can be obtained by the global lifts, that is, the intersection with the hypersurfaces H_s and \mathfrak{X} . In this case the Čech 1-cocycles are clearly 0, and since the coupling does not depend on the choice of lifts, the result must be identically 0.
- (3) However, it is important to consider local lifts which are not necessarily induced by the global lifts, in order to relate this argument to the case of deformations of degenerate rational curves.

The important point is that although the domain curves are different, the generator η of the obstruction class on $C_{0,s}$ and B_0 are identical after partial normalization of $C_{0,s}$ (see Remark 10). Namely, there is a natural one to one correspondence between the components of B_0 and C_0 , and the restrictions of the generators of the dual obstruction $H^0(B_0, \mathcal{N}_{\psi_0}^\vee \otimes \omega_{B_0})$ and $H^0(C_{0,s}, \mathcal{N}_{\varphi_{0,s}}^\vee \otimes \omega_{C_{0,s}})$ (these are both isomorphic to \mathbb{C} by Lemma 9) are precisely the same.

Moreover, in the calculation of the obstruction in Subsection 3.1, clearly it is possible to choose the covering $\{U_i\}$ of $C_{0,s}$ so that for any node of the image $\varphi_{0,s}(C_{0,s})$, there is only one U_i which contains the inverse image of that node. The covering $\{U_i\}$ gives rise to a covering $\{\tilde{U}_i\}$ of B_0 in an obvious way. That is, if $p : B_0 \rightarrow C_0$ is the partial normalization, then \tilde{U}_i is the inverse image $p^{-1}(U_i)$. Note that some \tilde{U}_i has 2 components.

Choosing the covering in this way, the Čech 1-cocycle defined by the local lifts on each covering is supported on open subsets of $C_{0,s}$ or B_0 which do not contain a node. When there is a local lift of $C_{0,0}$ on a neighborhood of each node such that it can also be seen as a local lift of B_0 , the following holds. Namely, if the cohomology class of the Čech 1-cocycle defined by the local lifts on $C_{0,0}$ is 0, then the cohomology class defined by the same Čech 1-cocycle on B_0 is also 0, since if the class is given as the coboundary of a Čech 0-cochain $\{\eta_i\}$ on $C_{0,0}$ associated to the covering $\{U_i\}$, it naturally determines a Čech 0-cochain on B_0 associated to the covering $\{\tilde{U}_i\}$, and its coboundary is the given Čech 1-cocycle on B_0 .

The condition that the cohomology class of the Čech 1-cocycle defined by the local lifts on $C_{0,0}$ is 0 can be assured when such lifts can be extended to a family of lifts on $C_{0,s}$, since the corresponding cohomology classes on $C_{0,s}$, $s \neq 0$ is 0 as we noted in Remark 11.

Based on this observation, we prove the following.

Proposition 12. *The map ψ_0 has smoothings up to any order.*

Proof. It suffices to show that the obstruction class in $H^1(B_0, \mathcal{N}_{\psi_0})$ defined as in Subsection 3.1 at each order of deformation vanishes. In turn, this is the same as showing the pairing between the obstruction class and the generator η of $H^0(B_0, \mathcal{N}_{\psi_0}^\vee \otimes \omega_{B_0})$ vanishes.

By the argument above, it suffices to show that there is a local lift of ψ_0 which can also be seen as a local lift of the degenerate genus 3 curve $\varphi_{0,0}$, which in turn is the specialization of the family of local lifts of $\varphi_{0,s}$.

Since $\mathfrak{X} \rightarrow \mathbb{C}$ is log smooth around a neighborhood of the image $\psi_0(B_0)$, and ψ_0 is a map between varieties with log structures, each point of B_0 has a suitable neighborhood on which ψ_0 has a lift. Let p be a node of $\psi_0(B_0)$ which is contained in the singular set \mathcal{S} of \mathfrak{X} and $\psi_{1,p} : U_{1,p} \rightarrow \mathfrak{X}$ be a local lift of ψ_0 around such a neighborhood of p . Here $U_{1,p}$ is a suitable lift of a neighborhood of the inverse image of p . Note that since ψ_0 is pre-smoothable, the inverse image of p is the set consisting of 2 regular points which belong to different components of B_0 . In particular, the neighbourhood $U_{1,p}$ has 2 connected components.

Recall that the set of local lifts is parameterized by the set of local sections of the normal sheaf \mathcal{N}_{ψ_0} of ψ_0 (up to translation by sections of the tangent sheaf Θ_{B_0}). According to Proposition 8, this sheaf, restricted to each component of $U_{1,p}$, is the sheaf of usual normal sheaf of a line in \mathbb{P}^2 , which is 0 at p . Using this freedom, we can change $\psi_{1,p}$ so that the image is contained in a hyperplane in $\mathbb{P}^3 \times \text{Spec} \mathbb{C}[t]/t^2$.

Then it is easy to extend this hyperplane to a family of hyperplanes parametrized by s , whose intersection with \mathfrak{X} gives a family of local lifts of genus 3 curves. This gives the required lift of ψ_0 around p . At the points which are not the intersection with \mathcal{S} , the analytic structure of B_0 and $C_{0,0}$ are the same, so local lifts of B_0 can clearly be regarded as local lifts of $C_{0,0}$. Thus, these local lifts can be regarded both as local lifts of $\varphi_{0,0}$ and also of ψ_0 . Then by the above argument, the obstruction to lift ψ_0 vanishes.

The calculation for the obstructions to higher order deformations is similar. If $\psi_k : B_k \rightarrow \mathfrak{X}$ is a k -th order smoothing of ψ_0 , then since it is a curve of degree 4 in $\mathfrak{X} \subset \mathbb{P}^3 \times \mathbb{C}$, there is a family of hyperplanes $H_{0,t}$ over $\mathbb{C}[t]/t^{k+1}$ such that

$$H_{0,t} \cap \mathfrak{X} = \psi_k(B_k).$$

This can also be seen as a family φ_k of nodal genus 3 curves. As above, since ψ_k is a morphism between varieties with log structures and \mathfrak{X} is log smooth, there are local lifts of ψ_k , and also we can assume that each lift is given by the intersection between \mathfrak{X} and a family of hyperplanes $H_{0,t}$ over $\mathbb{C}[t]/t^{k+2}$. This extends to a family of local lifts of genus 3 curves, and as before, the family of obstructions to lift these curves vanish by an obvious reason. Then, the obstruction to lift ψ_k vanishes too, as argued above. \square

Note that we can start from any general quartic polynomial f . This together with a standard result for algebraization [1], we have the following.

Corollary 13. *There is a Zariski open subset in the moduli space of quartic K3 surface whose members contain a rational curve of degree 4.* \square

Then by a suitable compactness result [7], we have the following.

Corollary 14. *Any quartic K3 surface contains a rational curve.* \square

This is obviously much weaker than the result proved in [15], which proved that every K3 surface has at least 1 rational curve. The merit of the method here is that the same idea can be promoted to produce infinitely many rational curves on continuous families of quartic K3 surfaces (in fact, more generally, of K3 surfaces which are complete intersection. See Section 7.1).

3.3. Explicit calculation of the obstruction. Although logically it is not necessary, in this section we perform some explicit calculation of the obstruction to lift maps such as $\varphi_{0,s}$ and ψ_0 considered in the previous subsection, based on [19]. We hope it would help the reader have some grasp of somewhat unconventional argument in the previous subsection. Explicitly, we first prove the following.

Proposition 15. *The obstruction to deform the map $\varphi_{0,s}$ ($s \neq 0$) to a map over $\mathbb{C}[t]/t^2$ vanishes.*

Remark 16. *Since the degenerate genus 3 curve $\varphi_{0,s}$ clearly deforms, we know that the obstruction should vanish without calculation. So this proposition is obvious and requires no proof. However, we carry out the calculation from the point of view of actually calculating the obstruction cohomology class from local data since it helps to understand the cases of rational curves later (see Subsection 3.3.1 below). Also, it may be interesting to see although the vanishing of the obstruction to lift $\varphi_{0,s}$ is a trivial fact, from the point of view of local computation, it is a consequence of rather elaborate cancelations.*

Proof. Using the homogeneous coordinates of \mathbb{P}^3 introduced at the beginning of Section 2, we write the hyperplane H by

$$\alpha x + \beta y + \gamma z + w = 0,$$

where our curve $\varphi_{0,s}(C_0)$ of degree 4 is given by the intersection between X_0 and H . Here α, β, γ are general complex numbers, so the hyperplane H does not intersect the locus \mathcal{S} .

The curve $\varphi_{0,s}(C_0)$ is the union of the following 4 lines:

$$l: \alpha x + \beta y + \gamma z = 0, \quad w = 0,$$

$$m: \beta y + \gamma z + \beta w = 0, \quad x = 0,$$

$$n: \alpha x + \gamma z + w = 0, \quad y = 0,$$

$$k: \alpha x + \beta y + w = 0, \quad z = 0.$$

Let us compute the contribution to the obstruction from the point p whose homogeneous coordinate is given by

$$[x, y, z, w] = [\beta, -\alpha, 0, 0].$$

This is the intersection of the lines l and k . The lines l and k contribute to the obstruction separately. First let us calculate the contribution from the line l . To do this, parametrize the line l by an affine parameter u as

$$l(u) = \left[1, -\frac{\alpha}{\beta} - \frac{\gamma}{\beta}u, u, 0 \right].$$

In particular, the parameter u can be taken as the pull back $\varphi_0^*\left(\frac{z}{x}\right)$. The line k is parametrized as well using an affine parameter v :

$$k(v) = \left[1, -\frac{\alpha}{\beta} - \frac{1}{\beta}v, 0, v \right].$$

Since the coordinate x is nonzero around the point p , we inhomogenize the given quartic polynomial f dividing by x^4 , and write the resulting polynomial in the form

$$\frac{1}{x^4}f(x, y, z, w) = c_0 + \left(\frac{y}{x} + \frac{\alpha}{\beta}\right) g_1\left(\frac{y}{x}\right) + \frac{z}{x} g_2\left(\frac{y}{x}, \frac{z}{x}\right) + \frac{w}{x} g_3\left(\frac{y}{x}, \frac{z}{x}, \frac{w}{x}\right).$$

Now we take a local lift of these parametrizations over $\mathbb{C}[t]/t^2$. A lift of $l(u) \setminus \{p\}$ has the following expression

$$\left(\frac{y}{x}, \frac{z}{x}, \frac{w}{x}\right) = \left(-\frac{\alpha}{\beta} - \frac{\gamma}{\beta}u + t\left(\frac{\varepsilon_1}{u} + a_1 + b_1u + c_1u^2 + \cdots\right), u, -t\beta\left(\frac{\varepsilon_1}{u} + a_1 + b_1u + c_1u^2 + \cdots\right)\right).$$

Remark 17. Note that if we take a global lift of $\varphi_{0,s}$, it is a quartic curve in \mathbb{P}^3 and so if it is on the family \mathfrak{X} , it must be on some hyperplane. Note also that it is a general principle that the obstruction cohomology class does not depend on the choice of local lifts using which the calculations are done. Thus, when we consider local lifts to calculate obstruction, we need not to restrict ourselves to choose particular local lifts which are contained in some hyperplanes. In the present case however, for notational simplicity, we choose the simplest one, that is, we choose local lifts which are contained in the fixed hyperplane to which the degenerate curve belongs.

Substitute these expressions to the equation

$$\frac{y}{x} \frac{z}{x} \frac{w}{x} + t \frac{1}{x^4} f(x, y, z, w) = 0$$

and calculate the condition for the vanishing of the coefficient of t . Then we obtain

$$\alpha\varepsilon_1 + c_0 = 0.$$

Similarly, the vanishing of the coefficient of tu implies

$$\alpha a_1 + \gamma\varepsilon_1 - \frac{\gamma}{\beta}g_1\left(-\frac{\alpha}{\beta}\right) + g_2\left(-\frac{\alpha}{\beta}, 0\right) = 0.$$

Combining these equations, we have

$$a_1 = \frac{\gamma}{\alpha^2}c_0 + \frac{\gamma}{\alpha\beta}g_1\left(-\frac{\alpha}{\beta}\right) - \frac{1}{\alpha}g_2\left(-\frac{\alpha}{\beta}, 0\right).$$

The obstruction is given by

$$\left(-\frac{\alpha}{\beta}\right)^{-1} a_1 = -\frac{\beta\gamma}{\alpha^3}c_0 - \frac{\gamma}{\alpha^2}g_1\left(-\frac{\alpha}{\beta}\right) + \frac{\beta}{\alpha^2}g_2\left(-\frac{\alpha}{\beta}, 0\right)$$

(see the proof of [19, Theorem 16]). Similarly, parameterizing a lift of the line $k(v) \setminus \{p\}$ in the form

$$\left(\frac{y}{x}, \frac{z}{x}, \frac{w}{x}\right) = \left(-\frac{\alpha}{\beta} - \frac{1}{\beta}v + t\left(\frac{\varepsilon'_1}{v} + a'_1 + b'_1v + c'_1v^2 + \cdots\right), -\frac{\beta}{\gamma} \cdot t\left(\frac{\varepsilon'_1}{v} + a'_1 + b'_1v + c'_1v^2 + \cdots\right), v\right),$$

the contribution from the curve k is given by

$$-\left(-\frac{\alpha}{\beta}\right)^{-1} a'_1 = \frac{\beta\gamma}{\alpha^3}c_0 + \frac{\gamma}{\alpha^2}g_1\left(-\frac{\alpha}{\beta}\right) - \frac{\beta\gamma}{\alpha^2}g_3\left(-\frac{\alpha}{\beta}, 0, 0\right).$$

The minus sign in front of the factor $\left(-\frac{\alpha}{\beta}\right)^{-1}$ is due to the relation

$$\frac{dz}{z} + \frac{dw}{w} = 0$$

for toric coordinates of \mathbb{P}^3 which are used to express the generator of the dual obstruction (the relation $(*)$ in the proof of Lemma 9). Summing the two obstructions up, the contribution to the obstruction from the point p is given by

$$\frac{\beta}{\alpha^2}g_2\left(-\frac{\alpha}{\beta}, 0\right) - \frac{\beta\gamma}{\alpha^2}g_3\left(-\frac{\alpha}{\beta}, 0, 0\right).$$

Now calculate the obstruction contributed from the point q given by

$$[x, y, z, w] = [-\gamma, 0, \alpha, 0].$$

This is the intersection of the lines l and n . In this case let us write the polynomial f in the form

$$\frac{1}{x^4}f(x, y, z, w) = d_0 + \left(\frac{z}{x} + \frac{\alpha}{\gamma}\right)h_1\left(\frac{z}{x}\right) + \frac{y}{x}h_2\left(\frac{y}{x}, \frac{z}{x}\right) + \frac{w}{x}h_3\left(\frac{y}{x}, \frac{z}{x}, \frac{w}{x}\right).$$

Calculating the obstructions contributed from the lines l and n as above, we see that the contribution from the line l is

$$\frac{\beta\gamma}{\alpha^3}d_0 + \frac{\beta}{\alpha^2}h_1\left(-\frac{\alpha}{\gamma}\right) - \frac{\gamma}{\alpha^2}h_2\left(0, -\frac{\alpha}{\gamma}\right).$$

The contribution from the line n is

$$-\frac{\beta\gamma}{\alpha^3}d_0 - \frac{\beta}{\alpha^2}h_1\left(-\frac{\alpha}{\gamma}\right) + \frac{\beta\gamma}{\alpha^2}h_3\left(0, -\frac{\alpha}{\gamma}, 0\right).$$

Summing these contributions, the obstruction contributed from the point q is

$$-\frac{\gamma}{\alpha^2}h_2\left(0, -\frac{\alpha}{\gamma}\right) + \frac{\beta\gamma}{\alpha^2}h_3\left(0, -\frac{\alpha}{\gamma}, 0\right).$$

Similarly, the contribution from the point

$$[x, y, z, w] = [0, -\gamma, \beta, 0],$$

which is the intersection of the lines l and m is given by

$$\frac{\gamma}{\beta^2}i_2\left(0, -\frac{\beta}{\gamma}\right) - \frac{\alpha\gamma}{\beta^2}i_3\left(0, -\frac{\beta}{\gamma}, 0\right),$$

where i_1, i_2, i_3 are polynomials defined by

$$\frac{1}{y^4}f(x, y, z, w) = e_0 + \left(\frac{z}{y} + \frac{\beta}{\gamma}\right)i_1\left(\frac{z}{y}\right) + \frac{x}{y}i_2\left(\frac{x}{y}, \frac{z}{y}\right) + \frac{w}{y}i_3\left(\frac{x}{y}, \frac{z}{y}, \frac{w}{y}\right).$$

The contributions to the obstruction from the other intersections are calculated similarly.

Now let us examine the relation between these contributions. First consider the sum $\frac{\beta}{\alpha^2}g_2\left(-\frac{\alpha}{\beta}, 0\right) - \frac{\beta\gamma}{\alpha^2}g_3\left(-\frac{\alpha}{\beta}, 0, 0\right)$ contributed from the point $l \cap k$. By definition, $\frac{z}{x}g_2\left(\frac{y}{x}, \frac{z}{x}\right)$ is the sum of monomials of $f(x, y, z, w)$ (factored by $\frac{1}{x^4}$) which has z as a factor but does not have w as a factor. Among these monomials, only the terms which are of degree 1 with respect to z contribute to $g_2\left(-\frac{\alpha}{\beta}, 0\right)$. These monomials of f are

$$x^3z, x^2yz, xy^2z, xz^3$$

times their coefficients.

Similarly, for $g_3\left(-\frac{\alpha}{\beta}, 0, 0\right)$, the monomials of f which are of degree 1 with respect to w and do not have z as a factor contribute. Thus, the monomials

$$x^3w, x^2yw, xy^2w, y^3w$$

of f (modulo coefficients) contribute.

Next consider contribution $\frac{\gamma}{\alpha^2}h_2(0, -\frac{\alpha}{\gamma}) - \frac{\beta\gamma}{\alpha^2}h_3(0, -\frac{\alpha}{\gamma}, 0)$ from the point $l \cap n$. By similar consideration, $h_2(0, -\frac{\alpha}{\gamma})$ has contributions from the monomials of $f(x, y, z, w)$ (factored by $\frac{1}{x^4}$) which have degree 1 with respect to y but do not have w as a factor. These monomials of f are

$$x^3y, x^2yz, xyz^2, yz^3$$

times their coefficients.

Similarly, for $h_3(0, -\frac{\alpha}{\gamma}, 0)$, monomials of f which are degree 1 with respect to w and do not have y as a factor contribute. Thus, the monomials

$$x^3w, x^2zw, xz^2w, z^3w$$

of f (modulo coefficients) contribute.

Also, to $\frac{\gamma}{\beta^2}i_2(0, -\frac{\beta}{\gamma}) - \frac{\alpha\gamma}{\beta^2}i_3(0, -\frac{\beta}{\gamma}, 0)$, the monomials

$$xy^3, xy^2z, xyz^2, xz^3$$

of f contribute to $i_2(0, -\frac{\beta}{\gamma})$, and the monomials

$$y^3w, y^2zw, yz^2w, z^3w$$

of f contribute to $i_3(0, -\frac{\beta}{\gamma}, 0)$.

Then consider the monomial x^3w of f (modulo coefficient). To the obstruction $\frac{\beta}{\alpha^2}g_2\left(-\frac{\alpha}{\beta}, 0\right) - \frac{\beta\gamma}{\alpha^2}g_3\left(-\frac{\alpha}{\beta}, 0, 0\right)$ from the point $l \cap k$, this term contributes

$$-\frac{\beta\gamma}{\alpha^2}$$

times the coefficient of the term x^3w in f . On the other hand, the contribution of this monomial to $-\frac{\gamma}{\alpha^2}h_2(0, -\frac{\alpha}{\gamma}) + \frac{\beta\gamma}{\alpha^2}h_3(0, -\frac{\alpha}{\gamma}, 0)$ is

$$\frac{\beta\gamma}{\alpha^2},$$

times the coefficient of the term x^3w of f . So these cancel each other.

Similarly, consider the monomial xy^2z . To the obstruction $-\frac{\beta^2}{\alpha^2}g_2\left(-\frac{\alpha}{\beta}, 0\right) + \frac{\beta}{\alpha^2}g_3\left(-\frac{\alpha}{\beta}, 0, 0\right)$, this term contributes

$$-\frac{\beta^2}{\alpha^2} \cdot \left(-\frac{\alpha}{\beta}\right)^2$$

times the coefficient of the term xy^2z of f , and to the obstruction $\frac{1}{\beta}i_2(0, \beta) - \frac{\alpha}{\beta^2}i_3(0, \beta, 0)$, it contributes

$$\frac{1}{\beta} \cdot \beta$$

times the coefficient of the term xy^2z of f . Thus, these again cancel each other.

Similarly, the contributions to the obstruction from various intersections of the lines k, l, m, n eventually sum up to zero. Thus the obstruction to lift the degenerate curve φ_0 to a curve over $\mathbb{C}[t]/t^2$ vanishes, as expected. \square

Although it becomes harder, given a curve $\varphi_{k,s}: C_{k,s} \rightarrow \mathfrak{X}$ over $\mathbb{C}[t]/t^{k+1}$ which specialize to the map $\varphi_{0,s}$ over $\mathbb{C}[t]/t$, the obstruction to lift it to a map over $\mathbb{C}[t]/t^{k+2}$ can be calculated in the same way. Of course it vanishes since we obviously have lifts of $\varphi_{0,s}$ up to any order. A basic but important point to notice is that given a k -th order lift $\varphi_{k,s}$ of the map $\varphi_{0,s}$, the obstruction to lift it to the next order is determined by the data of $\varphi_{k,s}$, and does not depend on which local lifts to take for the explicit calculation. In particular, if we have a family $\varphi_{0,s}$ of maps and its lift $\varphi_{k,s}$ parametrized analytically by a parameter s , then the obstruction classes to lift $\varphi_{k,s}$ to the next order is a cohomology valued analytic function of s .

3.3.1. Obstruction to lift degenerate rational curves. We make a notice about the properties of the local lifts of the maps $\varphi_{0,0}$ and ψ_0 at the intersection with the singular locus of \mathfrak{X} , from the point of view of the calculation above. As we mentioned in Section 2, there are coordinates X, Y, Z, t such that \mathfrak{X} is isomorphic to a neighborhood of the origin of the variety defined by

$$\{(X, Y, Z, t) \mid XY + tZ = 0\}.$$

The branches of the image of the curve $\varphi_{0,0}$ (and also of ψ_0) can be parametrized as

$$(X, Y, Z) = (u, 0, p_1u + p_2u^2 + \cdots), \quad t = 0,$$

$$(X, Y, Z) = (0, v, q_1v + q_2v^2 + \cdots), \quad t = 0,$$

respectively. Here u and v are suitable coordinates of the branches of C_0 (or components of B_0), and p_i, q_i are constants where $p_1q_1 \neq 0$.

One calculates that a lift of $\varphi_{0,0}$ or ψ_0 is given by the parametrization of the form

$$(X, Y, Z) = (u, t(-p_1 - p_2u - \cdots), p_1u + p_2u^2 + \cdots + t(r_0 + r_1u + r_2u^2 + \cdots)),$$

$$(X, Y, Z) = (t(-q_1 - q_2v - \cdots), v, q_1v + q_2v^2 + \cdots + t(r_0 + s_1v + s_2v^2 + \cdots)),$$

where r_i, s_i are constants. Among these local lifts, lifts of ψ_0 compatible with the log structure are those where the constant r_0 is 0. In fact, although all these first order lifts still have nodes at the point $u = v = 0$, one can easily check that the next order lift smoothes the node precisely when the constant r_0 is not 0.

Compared to the local lifts

$$\left(\frac{y}{x}, \frac{z}{x}, \frac{w}{x}\right) = \left(-\frac{\alpha}{\beta} - \frac{\gamma}{\beta}u + t\left(\frac{\varepsilon_1}{u} + a_1 + b_1u + c_1u^2 + \cdots\right), \quad u, \quad -t\beta\left(\frac{\varepsilon_1}{u} + a_1 + b_1u + c_1u^2 + \cdots\right)\right)$$

of $\varphi_{0,s}$ considered above, recall that the constant ε_1 satisfied the relation

$$\alpha\varepsilon_1 + c_0 = 0$$

where c_0 is the value of the (unhomogenized) defining equation f of the $K3$ surface at the image of the node.

This constant c_0 equals 0 if and only if the corresponding point on X_0 is the singularity of the total space \mathfrak{X} . Thus, the above expression for $\left(\frac{y}{x}, \frac{z}{x}, \frac{w}{x}\right)$ gives an explicit family of local lifts of the degenerate curves $\varphi_{0,s}$ including $s = 0$. Choosing other coefficients appropriately, we obtain a family of local lifts whose restriction to the parameter $s = 0$

can also be regarded as a lift of ψ_0 , which was used for the calculation of the obstruction to lift ψ_0 in the proof of Proposition 12. Since these local lifts are analytic, clearly their pairings with the generator of the dual obstruction class (see Lemma 9) give an analytic function with respect to the parameter of the family, as noted in the proof of Proposition 12.

Remark 18. *The above calculation shows that the "normal sheaf" of the curve $\varphi_{0,0}$ should be isomorphic to those of $\varphi_{0,s}$, $s \neq 0$ described in Proposition 8. However, since the curve $\varphi_{0,0}$ does not seem to fit in the formalism of log smooth deformation theory, we did not specify what the appropriate notion of the normal sheaf of $\varphi_{0,0}$ should be, and instead we chose the roundabout argument using auxiliary curves $\varphi_{0,s}$. In particular, we did not specify what the obstruction to lift the curve $\varphi_{0,0}$ is. If there is an appropriate formalism extending the log smooth deformation theory which is applicable to the curve $\varphi_{0,0}$, the vanishing of the obstruction to lift ψ_0 will follow directly from the same statement for the curve $\varphi_{0,0}$. Namely, since in this case we can talk about the obstruction to lift the curve $\varphi_{0,0}$, and its dual space should contain the generator calculated in Lemma 9 (but the actual (dual) obstruction space could be larger). Then its vanishing (which should be obvious since $\varphi_{0,0}$ deforms as in the case of $\varphi_{0,s}$) implies the vanishing of the obstruction to deform ψ_0 .*

4. CONSTRUCTION OF DEGENERATE RATIONAL CURVES

Now we turn to the construction of infinitely many rational curves on generic quartic K3 surfaces. The idea is the same as the previous section. Namely:

- Construct a degenerate curve on the special fiber X_0 of the degeneration \mathfrak{X} . In general, such a curve is seen as a degeneration of smooth curves of high genus. But there are some particular curves which are seen as degenerations of nodal rational curves.
- Compute the obstruction cohomology classes to deform these degenerate curves. An important point is that although the genus is different in general, the obstruction classes can be identified in a natural way (see Remark 10).
- Compute the actual obstruction. It is possible to directly compute it for the first order deformation, but in general it will be hard, if possible in principle. However, in the case of higher genus curves, one sees that the obstruction automatically vanishes since one can construct actual families of curves which degenerate to the given degenerate curves. Then the case of a degenerate rational curve is a limit of them, and since the obstruction can be seen as an analytic function of suitable parameters which is identically 0 when these parameters are nonzero, the obstruction vanishes also in the case of the rational curve.

In this section we carry out the first step. Namely, we construct degenerate curves on the special fiber. The second step will be done in Section 5 and the last step will be done in Section 6.

The construction of degenerate curves in the case of higher genus curves is simple. If we want a degeneration of general curves of degree $4m$, then take a product

$$\prod_{i=1}^m (a_i x + b_i y + c_i z + d_i w)$$

of m general linear functions. This gives a degenerate curve on X_0 which is a nodal union of $4m$ rational curves. Then take a general deformation of this polynomial over $\mathbb{C}[t]$, and consider the intersection of its zero and \mathfrak{X} . In this way one obtains a family of smooth curves which degenerates to the above degenerate curve.

Now we turn to the construction of degenerate rational curves. We start from the degree 4 rational curve $\psi_0 : C_0 \rightarrow X_0$ in the previous section. We choose a node, see Figure 5. Note that in the case of rational curves, we think that the intersection points between the curve and the singular locus of \mathfrak{X} are not nodes of the curve, see Figure 3. So the chosen point is away from the singular locus of \mathfrak{X} .

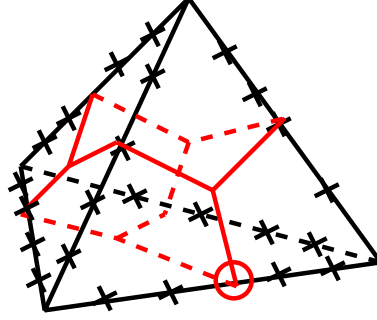


FIGURE 5. The node circled is chosen.

Then consider another degenerate curve $\sigma_0 : D_0 \rightarrow X_0$, whose image shares the chosen node with $\psi_0(C_0)$, and intersects the singular locus of \mathfrak{X} at 2 points, see Figure 6.

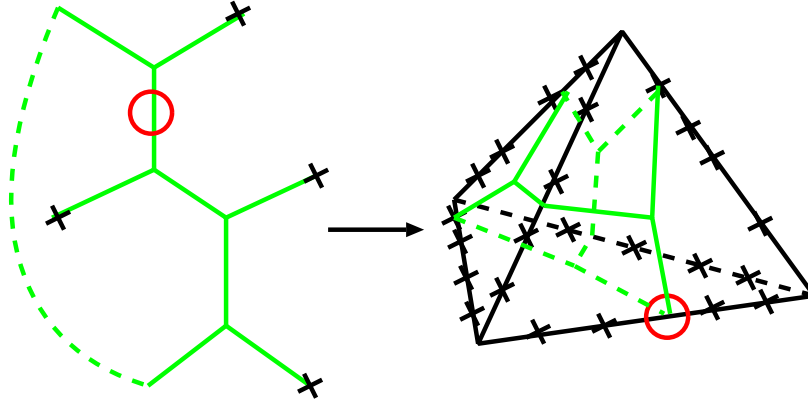


FIGURE 6. Tropicalized picture of a degenerate genus 1 curve on X_0 and its domain D_0 of the map. The ends of the edges connected by the dotted curve are glued so these edges are merged into 1 edge, which corresponds to a node of the holomorphic curve.

This is a degenerate genus 1 curve. Then take a finite covering of the domain curve D_0 . In the tropicalized picture, this corresponds to cut the loop of the graph and graft some copies of it, and make a loop again, see Figure 7. On the side of holomorphic curves, this is again a degenerate genus 1 curve.

Let us write by \tilde{D}_0 the resulting genus 1 curve. To obtain a degenerate rational curve, we cut the curve \tilde{D}_0 at 2 of its intersection with the degenerate rational curve C_0 at the

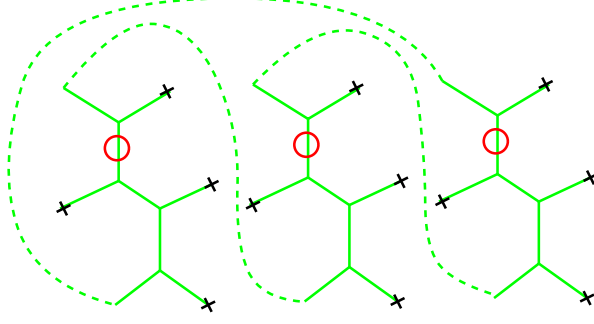


FIGURE 7. 3-fold covering of D_0 . The edges connected by dotted curves are glued into 1 edge, each of which corresponds to a node on the side of holomorphic curves. The nodes corresponding to the edges marked by circles intersect the degenerate rational curve.

chosen node. This divides the curve \tilde{D}_0 into 2 pieces. Then similarly cut the degenerate rational curve at the corresponding node, and graft parts of the degenerate rational curve to one of the components of the divided \tilde{D}_0 , see Figure 8, 9 and 10.

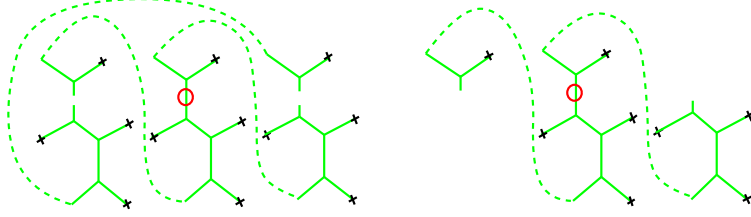


FIGURE 8. Cut \tilde{D}_0 at 2 nodes (picture on the left), and pick one of the resulting connected components (picture on the right).

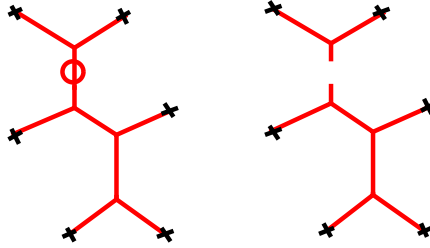


FIGURE 9. Similarly, cut the degenerate rational curve into 2 pieces.

When the degree of the covering of D_0 we first take is r , we write the resulting rational curve by this process by $D_{0,r}$. We have an obvious map

$$\psi_{0,r} : D_{0,r} \rightarrow X_0$$

induced from φ_0 and σ_0 . The image of $\psi_{0,r}$ is a curve of degree $4r$.

Remark 19. *There are many other possibilities to construct degenerate rational curves. Obviously we can choose other degenerate rational curves of degree 4, and degenerate*

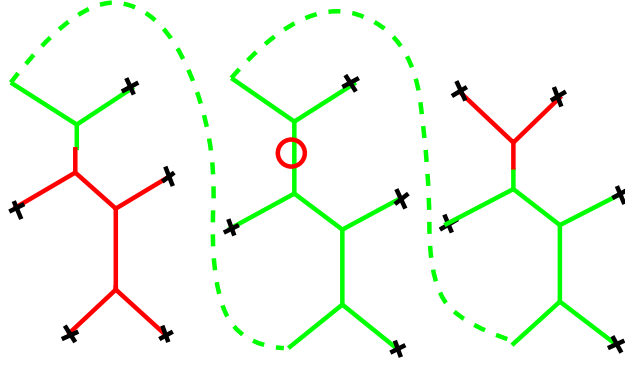


FIGURE 10. Graft the pieces of the rational curve to a part of divided \tilde{D}_0 . The result is a degenerate rational curve $D_{0,r}$.

elliptic curves of degree 4. Also, we can choose the position of the node they intersect. Furthermore, we can take a curve such that

- it intersects the degenerate rational curve at 2 nodes, and
- it intersects the singular locus of \mathfrak{X} at 1 point,

instead of the elliptic curve $\sigma_0 : D_0 \rightarrow X_0$. In this case, although the curve has genus 2, we can get a rational curve by a similar grafting process.

5. DUAL OF THE OBSTRUCTION COHOMOLOGY CLASSES OF DEGENERATE CURVES

In this section, we calculate the obstruction cohomology classes for the degenerate rational curve $\psi_{0,r} : D_{0,r} \rightarrow X_0$ constructed in the previous section, as well as for degenerate higher genus curves.

5.1. Dual obstruction classes of degenerate rational curves. Let

$$\psi_{0,r} : D_{0,r} \rightarrow X_0$$

be a degenerate rational curve of degree $4r$ as above. Note that the domain $D_{0,r}$ of $\psi_{0,r}$ is a union of smooth rational curves

$$D_{0,r} = \cup_{i=1}^{4r} D_{0,r,i},$$

here $D_{0,r,i}$ is an irreducible component. The curve $D_{0,r}$ has $4r + 2$ marked points on its smooth locus which are mapped to the singular locus of \mathfrak{X} .

The curve $D_{0,r}$ and an open neighborhood of the image $\psi_{0,r}(D_{0,r})$ in \mathfrak{X} have natural log structures such that the composition of the map $\psi_{0,r}$ and the projection $\mathfrak{X} \rightarrow \mathbb{C}$ can be equipped with a structure of a log smooth morphism. Let $\mathcal{N}_{\psi_{0,r}}$ be the log normal sheaf of the map ψ_0 as a map between log schemes. Let

$$H = H^1(D_{0,r}, \mathcal{N}_{\psi_{0,r}})^\vee \cong H^0(D_{0,r}, \mathcal{N}_{\psi_{0,r}}^\vee \otimes \omega_{D_{0,r}})$$

be the dual space of the cohomology group $H^1(D_{0,r}, \mathcal{N}_{\psi_{0,r}})$ in which the obstruction to lift $\psi_{0,r}$ lies. Here $\omega_{D_{0,r}}$ is the dualizing sheaf of $D_{0,r}$.

In this subsection, we prove the following.

Lemma 20. *The dual obstruction space H has dimension 1.*

Proof. This is just a straightforward extension of Lemma 9. Namely, on each component of $D_{0,r}$, sections of the restriction of the sheaf $\mathcal{N}_{\psi_{0,r}}^\vee \otimes \omega_{D_{0,r}}$ to it can be described as in the proof of Lemma 9. In particular, it is 1 dimensional and the value of the residue at one of the intersections with the toric divisors determines the section. Then a section on one component can be uniquely extended to the whole curve by the compatibility at the nodes. This proves the lemma. \square

5.2. Dual obstruction classes of degenerate higher genus curves. Now we calculate the (dual) obstruction class for degenerate general curves. Recall that such a family of curves is obtained as the intersection of the set of zeroes of

$$tg + \prod_{i=1}^r (a_i x + b_i y + c_i z + d_i w),$$

where g is a general homogeneous polynomial of degree r , and \mathfrak{X} . This family can be seen as a map $\Phi : \mathcal{C} \rightarrow \mathfrak{X}$, which in fact is an inclusion. Here \mathcal{C} is a flat family of curves whose general fiber is an irreducible nonsingular curve and the central fiber is a nodal union of nonsingular rational curves.

Let C_s be the fiber of \mathcal{C} over $s \in \mathbb{C}$ and let $\varphi_s : C_s \rightarrow \mathfrak{X}$ be the restriction of Φ . The curve C_0 has a natural log structure as described in Section 2 such that the composition of the map φ_0 and the projection $\mathfrak{X} \rightarrow \mathbb{C}$ can be equipped with a structure of a log smooth morphism. The obstruction space of the map φ_0 is given by

$$Ext_{C_0}^2(\varphi_0^* \Omega_{X_0} \rightarrow \Omega_{C_0}, \mathcal{O}_{C_0}),$$

see [12, Proposition 3.14].

Lemma 21. *The space $Ext_{C_0}^2(\varphi_0^* \Omega_{X_0} \rightarrow \Omega_{C_0}, \mathcal{O}_{C_0})$ is 1 dimensional.*

Proof. In the present case, since the image of φ_0 is a complete intersection, the kernel $\mathcal{I}/\mathcal{I}^2$ of the complex

$$\varphi_0^* \Omega_{X_0} \rightarrow \Omega_{C_0} \rightarrow 0,$$

is locally free of rank 1 (see [8, Lemma 27.5]). Here \mathcal{I} is the ideal sheaf of the image of φ_0 .

The sheaf $\mathcal{I}/\mathcal{I}^2$ is described as follows. It suffices to describe it restricted to each irreducible component of C_0 . A component $C_{0,v}$ of C_0 mapped to \mathbb{P}_i^2 by φ_0 has several special points. Namely,

- Nodes of C_0 mapped to the interior of \mathbb{P}_i^2 . Let us write them by $\{p_i\}$
- 3 nodes of C_0 mapped to the boundary of \mathbb{P}_i^2 . Let us write them by q_1, q_2, q_3 .

Then the restriction of $\mathcal{I}/\mathcal{I}^2$ to $C_{0,v}$ is isomorphic to

$$\mathcal{Q}(-\sum p_i),$$

here \mathcal{Q} is the usual conormal sheaf of a line in \mathbb{P}^2 , which is isomorphic to $\mathcal{O}(-1)$.

Then we have isomorphisms

$$\begin{aligned} Ext_{C_0}^2(\varphi_0^* \Omega_{X_0} \rightarrow \Omega_{C_0}, \mathcal{O}_{C_0}) &\cong H^1(C_0, (\mathcal{Q}(-\sum p_i))^\vee) \\ &\cong H^0(C_0, \mathcal{Q}(-\sum p_i) \otimes \omega_{C_0})^\vee, \end{aligned}$$

where ω_{C_0} is the dualizing sheaf. The restriction of ω_{C_0} to the component $C_{0,v}$ is the sheaf of meromorphic differential forms which can have poles at $\{p_i\}$ and $\{q_j\}$. Thus, the sheaf

$\mathcal{Q}(-\sum p_i) \otimes \omega_{C_0}$ on $C_{0,v}$ is in fact isomorphic to \mathcal{N} which we considered in the case of rational curves. In particular, the space of sections is 1 dimensional, and generated by the meromorphic 1-form described in the proof of Lemma 9. From this, it follows that the cohomology group $H^1(C_0, \mathcal{I}/\mathcal{I}^2) \cong \text{Ext}_{C_0}^2(\varphi_0^* \Omega_{X_0} \rightarrow \Omega_{C_0}, \mathcal{O}_{C_0})$ is 1 dimensional. \square

6. VANISHING OF OBSTRUCTION

In this section we finish the proof of the following theorem.

Theorem 22. *The moduli space of quartic K3 surfaces contains a Zariski open subset whose members contain infinitely many irreducible rational curves.*

Proof. The proof of Lemma 21 gives an explicit description of a generator η of the dual of the obstruction cohomology class. On the other hand, the obstruction cohomology class ν is given by the Čech 1-cocycle obtained as the difference of local lifts of φ_0 on a suitable covering of C_0 as in Subsection 3.1. Since φ_0 can be deformed by construction, this obstruction class clearly vanishes.

On the other hand, the degenerate rational curve constructed in Section 4 is presented by defining equations of the form

$$(ax + by + cz + dw) \cdot (\alpha x + \beta y + \gamma z + \delta w)^{r-1} = 0, \quad xyzw = 0,$$

where the hyperplane defined by $ax + by + cz + dw = 0$ intersects 3 singular points of the total space \mathfrak{X} (see Figure 5), while the hyperplane defined by $\alpha x + \beta y + \gamma z + \delta = 0$ intersects 2 singular points (see Figure 6). The line which is the intersection of these hyperplanes intersects the singular locus of X_0 (not of \mathfrak{X}) at one point (the circled nodes of Figure 5 and Figure 6).

We introduce another parameter s to the family of the degenerating higher genus curve:

$$tg + \prod_{i=1}^r (a_i(s)x + b_i(s)y + c_i(s) + z + d_i(s)w) = 0$$

so that the part $\prod_{i=1}^r (a_i(s)x + b_i(s)y + c_i(s) + z + d_i(s)w)$ converges to $(ax + by + cz + dw) \cdot (\alpha x + \beta y + \gamma z + \delta w)^{r-1}$ when $s \rightarrow 0$. The coefficients $a_i(s), b_i(s), c_i(s), d_i(s)$ are taken general so that the zero locus of $\prod_{i=1}^r (a_i(s)x + b_i(s)y + c_i(s) + z + d_i(s)w)$ does not intersect the singular locus of \mathfrak{X} when $s \neq 0$. Then as we noted above, the obstruction of the degenerate curve to lift it to a higher genus curve vanishes (when $s \neq 0$).

Now the coupling of the Čech 1-cocycle obtained from the local lifts of the degenerate higher genus curves which represent the obstruction class with the generator η of the dual obstruction class is an analytic function of s defined on a punctured disk around the origin of the complex plane. By the above observation, this function is identically 0. As in the proof of Proposition 12 (see also Subsection 3.3.1), these local lifts can be taken so that the lifts of the curve at $s = 0$ can also be seen as lifts of the rational curve $\psi_{0,r}$ defined in Subsection 5.1, compatible with the log structure. Then by the identification of the dual obstruction classes between the degenerate higher genus curve and the degenerate rational curve (see the proofs of Lemmas 20 and 21), the obstruction to lift the degenerate rational curve, which is given by the limit $s \rightarrow 0$ of the above analytic function, also vanishes. Thus, the degenerate rational curve has a first order lift.

Now since the degenerate rational curve is a complete intersection, its deformation is also a complete intersection. Thus, the lifted rational curve can also be presented as

the intersection of \mathfrak{X} and a suitable hypersurface of degree r . On the other hand, this hypersurface can be extended to a family parametrized by s such that the intersection of its general member and \mathfrak{X} is a smooth curve with higher genus. As before, the obstructions to lift these smooth curves vanish. Then the obstruction to lift the rational curve must also vanish by the same argument as above. This proves the existence of the lift of the degenerate rational curve to any order. Then a suitable algebraization theorem [1] assures the existence of a rational curve on the smooth $K3$ surface (in fact, since rational curves are rigid, the formal deformation coincides with the actual one).

Also, the validity of the construction does not depend on the choice of the degenerate rational curve (in particular, its degree), and the construction is valid on the region where the coordinates on the total space introduced in Subsection 2.2 are valid. This proves the theorem. \square

7. FURTHER TOPICS

7.1. Other $K3$ surfaces. The argument in the main text extends straightforwardly to other $K3$ surfaces which are complete intersections. For example, for the intersection of a generic quadric hypersurface and a generic cubic hypersurface in \mathbb{P}^4 defined by quadric and cubic homogeneous polynomials f and g is a $K3$ surface. Then consider the deformations of these polynomials $x_0x_1 + tf$ and $x_2x_3x_4 + tg$, where x_i are homogeneous coordinates of \mathbb{P}^4 and $t \in \mathbb{C}$ is the parameter for degeneration. Then the intersection

$$\{x_0x_1 + tf = 0\} \cap \{x_2x_3x_4 + tg = 0\}$$

gives a degenerating family of $K3$ surfaces whose special fiber is the union of 6 projective planes whose intersection polytope is given by the triangular dipyrmaid, see Figure 11. The singular set of the total space \mathfrak{X} of the degeneration are 24 points, marked in the picture. Choosing a hyperplane so that it intersects 4 of these singular points, we obtain a degenerate rational curve. This degenerate rational curve is in turn the limit of degenerate genus 4 curves, whose obstruction to lift obviously vanish. Then the same argument in the main text shows that the obstruction to lift the degenerate rational curve also vanishes.

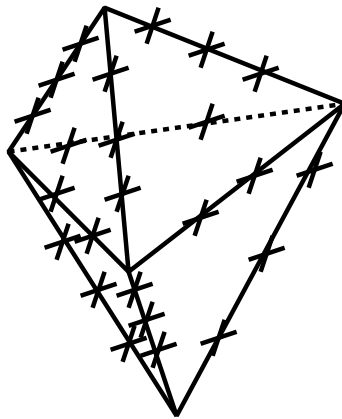


FIGURE 11.

Similarly, for the intersection of 3 generic quadratic hypersurface in \mathbb{P}^5 , the existence of infinitely many irreducible rational curve can be proved. In this case, the relevant

intersection polytope of the degenerate $K3$ surface is the octahedron and the total space has 24 singular points again.

More generally, we can consider complete intersection $K3$ surfaces in weighted projective spaces. In the cases of codimension 1 or 2, they are classified in [6]. See also [23]. Although the author has not checked except a few cases, the same method in this paper should be able to be applied to these $K3$ surfaces.

For example, consider a degeneration of a hypersurface of degree 5 in $\mathbb{P}(1, 1, 1, 2)$. If we write the homogeneous coordinates of $\mathbb{P}(1, 1, 1, 2)$ by x_0, x_1, x_2, x_3 , such a degeneration \mathfrak{X} is given by

$$x_0x_1x_2x_3 + tf = 0,$$

where f is a general weighted homogeneous polynomial of degree 5. The central fiber X_0 of \mathfrak{X} is the union of 1 \mathbb{P}^2 and 3 $\mathbb{P}(1, 1, 2)$. The intersection complex of the central fiber is given as follows:

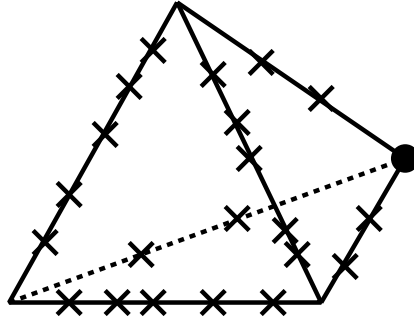


FIGURE 12. The intersection complex of the central fiber of a degeneration of a singular $K3$ hypersurface in $\mathbb{P}(1, 1, 1, 2)$. The bullet is the singular point of $\mathbb{P}(1, 1, 2)$ and the cross marks are the singular locus of the total space \mathfrak{X} (aside from the bullet).

Taking the intersection with a suitable hypersurface of degree 2, we have a degenerate rational curve.

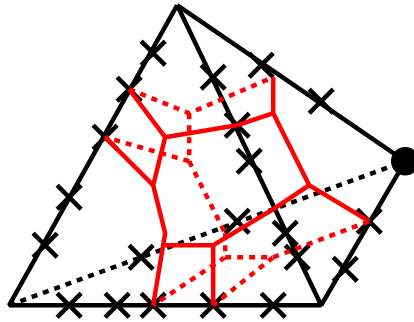


FIGURE 13.

Using such degenerate rational curves of low degree, we can construct infinitely many degenerate rational curves as before.

Another point which is different from the degeneration of complete intersection $K3$ surfaces in projective spaces is that general member will be usually singular. In the

case of degree 5 hypersurfaces in $\mathbb{P}(1, 1, 1, 2)$ above, a general member will have A_1 type singularity, which degenerates to the bullet in Figures 12 and 13. Accordingly, the number of moduli parameters of these K3 hypersurfaces is lower than 19, and to obtain the full moduli, we embed it to a higher dimensional projective space \mathbb{P}^N so that we can deform the singularity.

Then if we take the degree of degenerate rational curves to be sufficiently large, it will be realized as the intersection of X_0 and a suitable hypersurface in \mathbb{P}^N . Then the argument in the main text applies, so that we can produce infinitely many rational curves on Zariski general K3 surfaces in this family.

Although the author has not proved that this method can be applied to all the complete intersections in weighted projective spaces, as far as he checked, it seems to work well in each case (and it should work in the light of the denseness of rational curves on K3 surfaces mentioned later in Subsection 7.3). However, in these cases many rational curves will degenerate to a curve which intersects the fixed points of the torus action (that is, the vertices of the intersection complex of the central fiber of the degeneration), which require more work to handle with.

Also, since there are nice degenerations of general K3 surfaces [5, 3], it would be quite plausible that the method of this paper can be applied to produce rational curves on more general types of K3 surfaces by looking closely at these degenerations.

In fact, we need not restrict ourselves to K3 surfaces, and the same strategy will work in any situation where we can construct a nice degeneration. Examples of these are given in [19] for the case of rational curves in quintic Calabi-Yau 3 folds and families of holomorphic discs in hypersurfaces of any degree, in [20] for the case of rational curves on Fano complete intersections, and in [22] for the case of holomorphic curves on abelian surfaces.

7.2. Construction of other curves. As we noted in Remark 10, one of the good points of the construction in this paper is that the calculation of the obstruction hardly depends on the genus of curves. In particular, if we start from suitable degenerate curves, we can produce various kind of curves on Zariski general quartic K3 surfaces.

As an example, we consider the following construction. Take the degenerate rational curve in Figure 5 and a degenerate genus 2 curve as in the following Figure 14, which intersects the degenerate rational curve at the circled node.

Then cut the domain curve at the circled node and graft r copies of the resulting curve, and finally graft the parts of the cut degenerate rational curve in the same way as in Figures 9 and 10. See figures 15, 16 and 17.

This gives a degenerate curve of genus r . By the same argument as in the previous section, this curve can be smoothed. Thus, we have the following.

Proposition 23. *There is a Zariski dense subset in the moduli space of quartic K3 surfaces whose members contain irreducible curves of any genus.* \square

In fact, the choice of the degenerate genus 2 curve has 1 dimensional freedom, so each curve of genus r comes with an r dimensional family.

7.3. Denseness of rational curves and the solution to full Conjecture 1. The argument in this paper produced infinitely many irreducible rational curves on Zariski

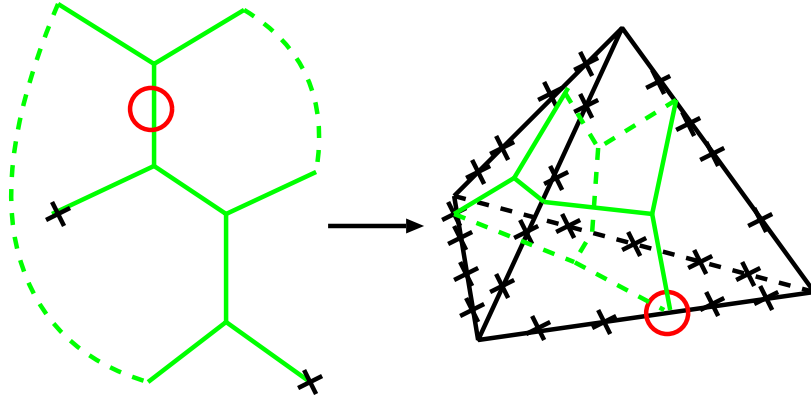


FIGURE 14. Tropicalized picture of a degenerate genus 2 curve on X_0 and its domain D_0 of the map. The ends of the edges connected by the dotted curve are glued.

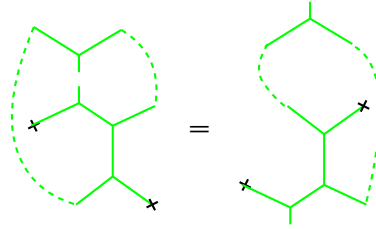


FIGURE 15.

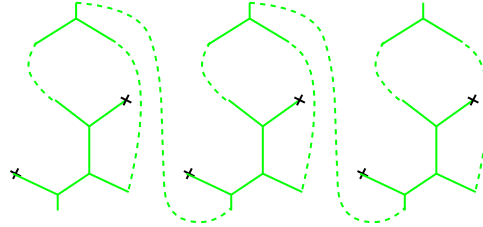


FIGURE 16.

general quartic K3 surfaces. But Conjecture 1 states that not only Zariski dense but *all* K3 surfaces would have infinitely many rational curves.

On the other hand, there is another conjecture concerning rational curves on K3 surfaces.

Conjecture 24. Let X be any K3 surface over complex numbers. Then the union of all the rational curves is dense in X with respect to the classical topology.

A remarkable result in this direction is proved by Chen and Lewis [4], which states that for a general polarized K3 surface, the union of all the rational curves is dense with respect to the classical topology.

By expanding the argument in this paper, we will be able to show that there is a Zariski dense open subset of the moduli space of quartic K3 surfaces whose members satisfy the conclusion of Conjecture 24. Namely, by using the flexibility of the construction

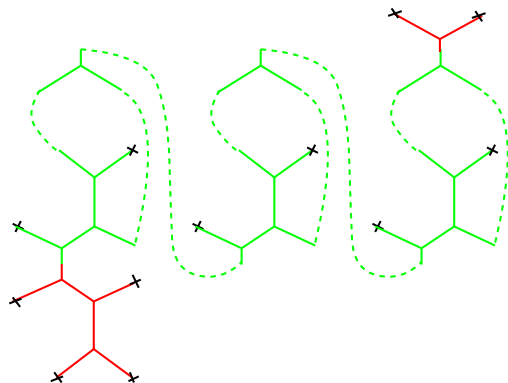


FIGURE 17.

mentioned in Remark 19, we can produce plenty of degenerate rational curves which are all smoothable. In fact, one can produce so many rational curves that their intersection with each of ℓ_i (the intersection of the 2 neighboring components of X_0) will be dense. Then it follows that the union of the degenerate curves as well as the smoothed curves on the smoothings of X_0 will be dense in the classical topology.

Once this is proved, then by a suitable compactness theorem [7], the union of rational curves is dense in *all* quartic K3 surfaces. Then this implies Conjecture 1. We will give details of these arguments in the forthcoming paper.

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